

# Searching on Trees with Noisy Memory

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## Abstract

We consider a search problem on trees using unreliable guiding instructions. Specifically, an agent starts a search at the root of a tree aiming to find a treasure hidden at one of the nodes by an adversary. Each visited node holds information, called *advice*, regarding the most promising neighbor to continue the search. However, the memory holding this information may not be reliable. Modelling this scenario, we focus on a probabilistic setting. That is, the advice at a node is a pointer to one of its neighbors. With probability  $q$  each node is *faulty*, independently of other nodes, in which case its advice points at an arbitrary neighbor, chosen u.a.r. Otherwise, the node is *sound* and necessarily points at the correct neighbor. Crucially, the advice is *permanent*, in the sense that querying a node several times would yield the same answer. We evaluate the agent's efficiency by two measures: The *move complexity* denotes the expected number of edge traversals, and the *query complexity* denotes the expected number of queries.

Roughly speaking, the main message of this paper is that in order to obtain efficient search,  $1/\sqrt{\Delta}$  is a threshold for the *noise parameter*  $q$ . Essentially, we prove that above the threshold, every search algorithm has query complexity (and move complexity) which is both exponential in the depth  $d$  of the treasure and polynomial in the number of nodes  $n$ . Conversely, below the threshold, there exist an algorithm with move complexity  $\mathcal{O}(d\sqrt{\Delta})$ , and an algorithm with query complexity  $\mathcal{O}(\sqrt{\Delta} \log n)$ . For  $q$  that is below but close to the threshold, both these upper bounds are optimal.

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# 1 Introduction

Imagine driving a car in an unknown country knowing that a hurricane had just hit it and had turned some fraction of the road-signs. You are now faced with the situation that some road-signs are correct, while others are misleading. Can you still reach your destination fast?

This paper considers a basic search problem on trees, in which the goal is to find a treasure that is placed at one of the nodes by an adversary. We assume that each node holds information regarding which of its neighbors is closer to the treasure. However, this information, called *advice*, may be faulty with some probability. Many works consider noisy queries in the context of search, but it is typically assumed that queries can be resampled (see e.g., [3, 8, 9, 16]). In contrast, we assume that each location is associated with a single *permanent* advice. That is, faults are in the physical memory associated with the node, and hence querying the node again would yield the same answer. This difference is dramatic, as the search under our model does not allow for simple amplification procedures (similar to [5] albeit in the context of sorting).

This model is of course relevant to data-structure search, e.g., in binary search trees, but can also model searching using unreliable heuristics. This appears in the field of artificial intelligence, where it is beneficial to understand how to intelligently leverage weak heuristics in order to produce efficient search [19, 21]. This question is also meaningful outside the realm of classical computer science. Some of the authors, as a part of an interdisciplinary team, recently reported navigation relying on guiding instructions in the context of collaborative transport by ants [13]. There, a group of ants carry a large load of food aiming to transport it to their nest, while basing their navigation on unreliable<sup>1</sup> advice given by pheromones that are laid on the terrain. It was advocated in [13] that one of the mechanisms that allows ants to utilize the beneficial information in pheromones while avoiding deadlocks is a simple memoryless strategy, by which the group of carrying ants occasionally ignores the suggested advice and performs a random walk step instead.

## 1.1 The Noisy Advice Model

We start with some notation. Further notations are given in Section 1.4. Let  $T$  be an  $n$ -node tree<sup>2</sup> rooted at some arbitrary node  $\sigma$ . We consider an agent that is initially located at the root  $\sigma$  of  $T$ , aiming to find a node  $\tau$ , called the *treasure*, which is chosen by an adversary. The distance  $d(u, v)$  is the number of edges on the path between  $u$  and  $v$ . The depth of a node  $u$  is  $d(u) = d(\sigma, u)$ . Let  $d = d(\tau)$  denote the depth of  $\tau$ , and let the depth  $D$  of the tree be the maximal depth of a node. Finally, let  $\Delta_u$  denote the degree of node  $u$  and let  $\Delta$  denote the maximal degree in the tree.

Each node  $u \neq \tau$  is assumed to be provided with an *advice*, termed  $\text{adv}(u)$ , which provides information regarding the direction of the treasure. Specifically,  $\text{adv}(u)$  is a pointer to one of  $u$ 's neighbors. It is called *correct* if the pointed neighbor is one step closer to the treasure than  $u$  is. Each vertex  $u \neq \tau$  is *faulty* with probability  $q_u$ , in which case its advice points to one of its neighbors chosen uniformly at random<sup>3</sup> (and so possibly pointing at the correct one). Otherwise,  $u$  is considered *sound*, in which case its advice is correct. We call  $q_u$  the *noise parameter* of  $u$ , and define the *general noise parameter* as  $q = \max\{q_u \mid u \in T\}$ .

The agent can move by traversing edges of the tree. At any time, the agent can query its hosting node in order to “see” the corresponding advice and to detect whether the treasure is present there.

<sup>1</sup>Although the directions proposed by pheromones typically lead to the nest, trajectories as experienced by small ants may be inaccessible to the load, and hence directional cues left by ants sometimes lead the load towards dead-ends.

<sup>2</sup>We present the model for trees, but it should be clear that it can be similarly defined for arbitrary graphs.

<sup>3</sup>We also consider an adversarial variant, where this neighbor is chosen by an oblivious adversary.

The protocol terminates when the agent queries the treasure. We evaluate a search algorithm  $A$  by two measures: The move complexity, termed  $\mathcal{M}(A)$ , is the expected number of edge traversals, and the query complexity, termed  $\mathcal{Q}(A)$ , is the expected number of queries.

Expectation is taken over both the randomness involved in sampling advice and the possible probabilistic choices made by  $A$ . We note that when considering walking algorithms, we assume that the agent does not know the structure of the tree in advance, and discovers it as it moves. Conversely, when focusing on minimizing the query complexity only, we assume that the tree structure is known to the algorithm. Throughout the paper, we assume that algorithms can be probabilistic.

The noise parameters  $(q_u)_{u \in T}$  govern the accuracy of the environment. On the one extreme, if  $q_u = 0$  for all nodes, then advice is always correct. This case allows to find the treasure in  $d$  moves, by simply following each encountered advice. Alternatively, it also allows to find the treasure using  $\mathcal{O}(\log n)$  queries, by performing a separator based search. On the other extreme, if  $q_u = 1$  for all nodes, then advice is essentially meaningless, and the search cannot be expected to be efficient. An intriguing question is therefore to identify the largest noise parameter  $q$  that allows for efficient search. We evaluate the search measures with respect to the depth  $d$  of the treasure, the number of nodes  $n$ , and  $\Delta$ .

## 1.2 Our Results

Consider the noisy advice model on trees with maximum degree  $\Delta$  and depth  $D$ . Roughly speaking, we show that  $1/\sqrt{\Delta}$  is the threshold for the noise parameter  $q$ , in order to obtain efficient search. More precisely, we first establish (in Section 2 and Appendix A) the following lower bounds.

**Theorem 1.1.** *The following holds for any randomized algorithm  $A$  and any integer  $\Delta \geq 3$ .*

1. **Exponential complexity above the threshold.** *Consider a complete  $\Delta$ -ary tree. For every constant  $\varepsilon > 0$ , if  $q > \frac{1+\varepsilon}{\sqrt{\Delta}-1} \cdot (1 + \frac{1}{\Delta-1})$ , then both  $\mathcal{Q}(A)$  and  $\mathcal{M}(A)$  are exponential in  $D$ .*
2. **Lower bounds for any  $q$ .**
  - (a) *Consider a complete  $\Delta$ -ary tree. Then  $\mathcal{Q}(A) = \Omega(q\Delta \log_{\Delta} n)$ .*
  - (b) *For any integer  $d$ , there is a tree with at most  $d\Delta$  nodes, and a placement of the treasure at depth  $d$ , such that  $\mathcal{M}(A) = \Omega(dq\Delta)$ .*

Our main technical contribution is a walking algorithm (in Section 3) that is optimal up to a constant factor for the regime of noise below the threshold. Furthermore, this algorithm does not require prior knowledge of neither the tree's structure, nor the values of  $\Delta$ ,  $q$ ,  $d$ , or  $n$ . Adjusting this walking algorithm, we derive a query algorithm (in Section 4) that is optimal up to a factor of a constant times  $\log(\Delta)$  for the same regime of noise. Before stating our theorem, we need to following definition.

**Definition 1.2.** We say that Condition  $(\star)$  holds with parameter  $0 < \varepsilon < 1$  if for every node  $v$ :

$$q_v < \frac{1 - \varepsilon - \Delta_v^{-\frac{1}{4}}}{\sqrt{\Delta_v} + \Delta_v^{\frac{1}{4}}}.$$

**Theorem 1.3.** *There exist a deterministic walking algorithm  $A_{\text{walk}}$  and a deterministic query algorithm  $A_{\text{query}}$ , such that the following holds for any constant  $\varepsilon > 0$ . If Condition  $(\star)$  holds with parameter  $\varepsilon$  then: (1)  $\mathcal{M}(A_{\text{walk}}) = \mathcal{O}(\sqrt{\Delta}d)$ , and (2)  $\mathcal{Q}(A_{\text{query}}) = \mathcal{O}(\sqrt{\Delta} \log n)$ . The notation  $\mathcal{O}(\cdot)$  only hides a polynomial dependency on  $1/\varepsilon$ .*

First note that in Theorem 1.3, since  $\Delta_v \geq 2$ , then taking a small enough  $\varepsilon$  the condition is always satisfiable. Observe also that, taken together, Theorems 1.1 and 1.3 together with Condition  $(\star)$  imply that for every  $\varepsilon > 0$  and large enough  $\Delta$ , efficient search can be achieved if  $q < (1 - \varepsilon)/\sqrt{\Delta}$  but not if  $q > (1 + \varepsilon)/\sqrt{\Delta}$ .

Finally, we analyze the performance of simple memoryless algorithms called *probabilistic following*. At every step, the algorithm follows the advice at the current vertex with some fixed probability  $\lambda$ , and performs a random walk step otherwise. It turns out that such algorithms can perform well, but only in a very limited regime of noise. Specifically, when tuned properly, such an algorithm can achieve the optimal  $\mathcal{O}(d)$  move complexity provided that  $q < c_1/\Delta$ , for some positive constant  $c_1$ . On the other hand, it incurs exponential complexity where  $q > c_2/\Delta$ , for some other constant  $c_2$ . Interestingly, when  $q < c_1/\Delta$ , this algorithm works even in a semi-adversarial model where the faulty nodes are still chosen randomly, but an adversary specifies (beforehand) the behavior of faulty nodes. In this semi-adversarial variant, it turns out that probabilistic following algorithms are the best possible, as the threshold for efficient search, with respect to any algorithm, is roughly  $1/\Delta$ . Due to lack of space these results are discussed and proved in Appendix D.1.

### 1.3 Related Work

In computer science, search algorithms have been the focus of numerous works. Due to their importance, trees are particularly popular structures to investigate search, see e.g., [2, 17, 18, 20]. Within the literature on search, many works considered noisy queries [8, 9, 16], however, it was typically assumed that noise can be *resampled* at every query. As mentioned, dealing with permanent faults incurs challenges that are fundamentally different from those that arise when allowing queries to be resampled. To illustrate this difference, consider the simple example of a star graph and a constant  $q < 1$ . Straightforward amplification can detect the target in  $\mathcal{O}(1)$  expected number of queries. In contrast, in our model, it can be easily seen that  $\Omega(n)$  is a lower bound for both the move and the query complexities, for any constant noise parameter.

Sorting and searching with memory faults is the subject of [10–12]. In these works the faults can happen at any time during the execution of the algorithm. The algorithm knows their number, and its resilience should be to a worst case number of faults. A search problem on graphs in which the set of nodes with misleading advice is chosen by an adversary was studied in [14, 15], as part of the more general framework of the *liar models* [1, 4, 6, 22]. All these worst case models are however significantly different from the randomized one we consider, both in terms of techniques and of results. The subject of queries with probabilistic memory faults has been explicitly studied in the context of sorting [5], but it appears that the techniques therein are difficult to adjust to the context of graph searching.

The noisy advice model originated in the recent biologically centered work [13], aiming to abstract ant navigation that relies on unreliable guiding instructions. In that work, the authors modelled ant navigation as a probabilistic following algorithm, and noticed that an execution of such an algorithm can be viewed as an instance of Random Walks in Random Environment (RWRE) [7, 23]. Relying on results from this subfield of probability theory, the authors showed that when tuned properly, such algorithms enjoy linear move complexity on grids, provided that the bias towards the correct direction is sufficiently high.

### 1.4 Notations

For two nodes  $u, v$ , let  $\langle u, v \rangle$  denote the simple path connecting them, excluding the end nodes, and let  $[u, v] = \langle u, v \rangle \cup \{u\}$  and  $[u, v] = \langle u, v \rangle \cup \{v\}$ . For a node  $u$ , we denote by  $T(u)$  the subtree hanging down from  $u$ . We denote by  $\text{adv}(u)$  (resp.  $\overleftarrow{\text{adv}}(u)$ ) the set of nodes whose advice points

towards (resp. away from)  $u$ . By convention  $u \notin \overrightarrow{\text{adv}}(u) \cup \overleftarrow{\text{adv}}(u)$ . When used to describe only the advice seen so far in the algorithm,  $\text{adv}$  will be a partial function.

## 2 Lower Bounds

Our goal in this section is to prove items (1) and (2a) in Theorem 1.1. The remaining lower bound (2b) is proved in Appendix A.2.

### 2.1 Exponential Complexity Above the Threshold

We wish to prove Item (1) in Theorem 1.1. Namely, that for every fixed  $\varepsilon > 0$ , and for every complete  $\Delta$ -ary tree, if  $q \geq \frac{1+\varepsilon}{\sqrt{\Delta-1}} \cdot (1 + \frac{1}{\Delta-1})$ , then every randomized search algorithm has query (and move) complexity which is both exponential in the depth  $d$  of the treasure and polynomial in  $n$ . In fact, this lower bound holds even if the algorithm has access to the advice of all internal nodes.

We first quote a lemma that is proved in Appendix A.1:

**Lemma 2.1.** *Assume the treasure is placed in a leaf  $\tau$  of the complete  $\Delta$ -ary tree. Denote by  $\text{adv}$  the random advice on all internal nodes, then the expected number of leaves  $u$  satisfying  $|\overrightarrow{\text{adv}}(u)| > |\overrightarrow{\text{adv}}(\tau)|$ , is a lower bound on the query complexity of any algorithm.*

Using Lemma 2.1, all we need to do is approximate the number of leaves as stated above. When comparing the number of pointers that point towards each of two different nodes, only the pointers of the internal nodes on the path between them make a difference. The probability that a leaf  $u$  “beats” the treasure  $\tau$  in the sense of Lemma 2.1, is at least the probability that exactly one node on the path points to  $u$  and none of the rest point towards the treasure. This probability is at least  $\frac{q}{\Delta} \cdot (q \cdot (1 - \frac{1}{\Delta}))^{d(u,\tau)-2}$ . There are precisely  $(\Delta - 1)^D$  leaves whose distance from the treasure is  $2D$ . Therefore, the expected number of leaves that beat the treasure is at least:

$$\frac{q}{\Delta} (\Delta - 1)^D q^{2D-2} \cdot \left(1 - \frac{1}{\Delta}\right)^{2D-2} = \frac{\Delta}{q(\Delta - 1)^2} \cdot \left(\frac{q^2(\Delta - 1)^3}{\Delta^2}\right)^D.$$

As the condition of the theorem exactly implies that the term inside the parentheses is greater than 1, Item (1) in Theorem 1.1 follows.  $\square$

### 2.2 A Lower Bound of $\Omega(\sqrt{\Delta} \cdot \log_{\Delta} n)$ when $q \sim 1/\sqrt{\Delta}$

We now prove Item (2a) in Theorem 1.1. Specifically, we wish to prove that for  $\Delta \geq 3$ , on the complete  $\Delta$ -ary tree of depth  $D$ , any algorithm needs  $\Omega(q\Delta D)$  queries on expectation. Note that, in particular, when  $q$  is roughly  $1/\sqrt{\Delta}$ , and  $n$  is the tree size, the query complexity is  $\Omega(\sqrt{\Delta} \cdot \log_{\Delta} n)$ . Before proving this lower bound, we note that a straightforward application of Yao’s principle shows:

**Observation 2.2.** *Any randomized algorithm trying to find a treasure chosen uniformly at random between  $k$  identical objects will need an expected number of queries that is at least  $(k + 1)/2$ .*

To prove the lower bound of  $\Omega(q\Delta D)$ , consider the complete  $\Delta$ -ary tree of depth  $D$ . We prove by induction on  $D$ , that if the treasure is placed u.a.r. in one of the leaves, then the expected query complexity of any algorithm is at least  $q(\Delta/2 - 1)D$ . If  $D = 0$ , then there is nothing to show. Assume this is true for  $D$ , and we shall prove it for  $D + 1$ . Let  $T_1, \dots, T_{\Delta-1}$  be the subtrees hanging down from the root (in the induction, the “root” is actually an internal node, and so has  $\Delta - 1$  children), each having depth  $D$ . Let  $i$  be the index such that  $\tau \in T_i$ , and denote by  $Q$

the number of queries before the algorithm makes its first query in  $T_i$ . We will assume that the algorithm gets the advice in the root for free. Denote by  $Y$  the event that the root is faulty. In this case, Observation 2.2 applies, and we need at least  $\Delta/2 - 1$  queries to hit the correct tree. We subtracted one query from the count because we want to count the number of queries strictly before querying inside  $T_i$ . We therefore get  $\mathbf{E}[Q] \geq \Pr(Y) \cdot \mathbf{E}[Q | Y] \geq q(\Delta/2 - 1)$ . By linearity of expectation, using the induction hypothesis, we get the result for a uniformly placed treasure over the leaves, and so it holds also in the adversarial case. This concludes the proof.  $\square$

### 3 An Optimal Walking Algorithm

In this section we prove the first item in Theorem 1.3. At any given time, the walking algorithm needs to decide to which of the nodes on the border of the already discovered part it should continue. At first glance, they all look the same because the agent has no a priori knowledge of the structure of the whole tree. Therefore, the following algorithm is a natural first attempt.

**A simple attempt that fails.** Consider the simple greedy algorithm  $A_{fail}$  that goes at each step to the unvisited node having most arrows pointing to it among the neighbors of previously seen nodes. Unfortunately, as its name suggests, for most of the noise regime that interests us, this algorithm performs poorly. Indeed, consider a complete  $\Delta$ -ary tree, and assume that the treasure  $\tau$  is a child of the root  $\sigma$ . For any leaf  $v$  at depth  $D$ , which is not a descendant of the treasure, with probability  $\frac{q}{\Delta} \cdot q^{D-1}(1 - \frac{1}{\Delta})^{D-1}$  the root points towards  $v$  and all the rest of the nodes on the path  $\langle \sigma, v \rangle$  do not point towards the root. This makes all nodes of the path better ranked than the correct child of  $\sigma$ , which is  $\tau$  in this case, and so  $A_{fail}$  would visit them all before finding  $\tau$ . There are  $(\Delta - 1)^D$  leaves that are not descendants of the treasure, and hence the expected number of nodes visited at depth  $D$  before making the first step on the correct path is at least  $q^D(\Delta - 1)^{D-1}(1 - \frac{1}{\Delta})^D$  in expectation. For  $q > 3/\Delta$  (and in particular, when  $q$  is roughly  $1/\sqrt{\Delta}$ ) this number is exponential in the depth  $D$ .

#### 3.1 Algorithm Design following a Greedy Bayesian Approach

In our setting the treasure is placed by an adversary. However, we can still study algorithms induced by assuming that it is placed in one of the vertices according to some known distribution and see how they measure up in our worst case setting. This approach is similar to [3], which studies the closely related, yet much simpler problem of search on the line. Of course, the success of this scheme highly depends on the choice of the prior distribution we take.

To make our life easier, let us first assume that the structure of the tree is known to the algorithm. Also, we assume the treasure is placed in one of the leaves of the tree according to some known distribution  $\theta$ , and denote by  $\mathbf{adv}$  the advice on the nodes we have already visited. Aiming to find the treasure as fast as possible, a possible greedy algorithm explores the vertex that, given the advice seen so far, has the highest probability of having the treasure in its subtree.

We extend the definition of  $\theta$  to internal nodes by defining  $\theta(u)$  to be the sum of  $\theta(w)$  over all leaves  $w$  of  $T(u)$ . Given some  $u$  that was not visited yet, and given the previously seen advice  $\mathbf{adv}$ , the probability of the treasure being in  $u$ 's subtree  $T(u)$ , is:

$$\Pr(\tau \in T(u) | \mathbf{adv}) = \frac{\Pr(\tau \in T(u))}{\Pr(\mathbf{adv})} \Pr(\mathbf{adv} | \tau \in T(u)) = \frac{\theta(u)}{\Pr(\mathbf{adv})} \prod_{w \in \vec{\mathbf{adv}}(u)} \left( p_w + \frac{q_w}{\Delta_w} \right) \prod_{w \in \overleftarrow{\mathbf{adv}}(u)} \frac{q_w}{\Delta_w}.$$

Note that the advice seen so far is never for vertices in  $T(u)$  as we consider a walking algorithm, and  $u$  has not been visited yet. Therefore, if  $\tau \in T(u)$  then correct advice in  $\mathbf{adv}$  points to  $u$ . We ignore

the term  $p_w + q_w/\Delta_w$  as it is normally quite close to 1, and applying a log we can approximate the relative strength of a node by:

$$\log(\theta(u)) + \sum_{w \in \overleftarrow{\text{adv}}(u)} \log\left(\frac{q_w}{\Delta_w}\right).$$

We do not want to assume that our algorithm knows  $q_w$ , but we do assume that in the worst scenario  $q_w \sim 1/\sqrt{\Delta_w}$ . Assigning this value and rescaling we finally define:

$$\text{score}(u) = \frac{2}{3} \log(\theta(u)) - \sum_{w \in \overleftarrow{\text{adv}}(u)} \log(\Delta_w).$$

When comparing two specific vertices  $u$  and  $v$ ,  $\text{score}(u) > \text{score}(v)$  iff:

$$\sum_{w \in [u, v] \cap \overrightarrow{\text{adv}}(u)} \log(\Delta_w) - \sum_{w \in [u, v] \cap \overrightarrow{\text{adv}}(v)} \log(\Delta_w) > \frac{2}{3} \log\left(\frac{\theta(v)}{\theta(u)}\right).$$

This is because any advice that is not on the path between  $u$  and  $v$  contributes the same to both sides, as well as advice on vertices on the path that point sideways, and not towards  $u$  or  $v$ <sup>4</sup>. Since we use this score to compare two vertices that are neighbors of already explored vertices, and our algorithm is a walking algorithm, then we will always have all the advice on this path<sup>5</sup>. In particular, the answer to whether  $\text{score}(u) > \text{score}(v)$ , does not depend on the specific choices of the algorithm, and does not change throughout the execution of the algorithm, even though the scores themselves do change. The comparison depends only on the advice given by the environment.

Let us try and justify the score criterion at an intuitive level. Consider the case of a complete  $\Delta$ -ary tree, with  $\theta$  being the uniform distribution on the leaves<sup>6</sup>. Here  $\text{score}(u) > \text{score}(v)$  if (cheating a little by thinking of  $\log(\Delta)$  and  $\log(\Delta - 1)$  as equal):

$$|\overrightarrow{\text{adv}}(u) \cap [u, v]| - |\overrightarrow{\text{adv}}(v) \cap [u, v]| > \frac{2}{3}(d(u) - d(v)).$$

If, for example, we consider two vertices  $u, v \in T$  at the same depth, then  $\text{score}(u) > \text{score}(v)$  if there is more advice pointing towards  $u$  than towards  $v$ . If the vertices have different depths, then the one closer to the root has some advantage, but it can still be beaten.

For general trees, perhaps the most natural  $\theta$  to take is the uniform distribution on all nodes (or just on all leaves - this choice is actually similar). It is also a generalization of the example above. Unfortunately, however, while this works well on the complete  $\Delta$ -ary tree, we show in Appendix C that this approach fails in the general case.

### 3.2 Algorithm $A_{\text{walk}}$

In our context, there is no distribution over treasure location and we are free to choose  $\theta$  as we like. We take  $\theta$  to be the distribution defined by a simple random process. Starting at the root, at each

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<sup>4</sup>At this point, it is tempting to define  $\text{score}(u)$  as the sum of weighted advice from the root to  $u$ . However, when comparing two vertices, this will result in a double counting of the advice on their least common ancestor, which we would prefer to avoid.

<sup>5</sup>In our walking algorithm we will always compare two vertices that are yet unexplored, and so the path in the above sums is actually  $\langle u, v \rangle$ . However, in our query algorithm we will need to compare vertices that were already explored, and in that case, the comparison will include advice on the whole path  $[u, v]$ .

<sup>6</sup>Actually, a similar formula could be derived choosing  $\theta$  to be the uniform distribution over all nodes, but for technical reasons it is easier to restrict it to leaves only.

step, walk down to a child u.a.r. until reaching a leaf. For a leaf  $v$ , define  $\theta(v)$  as the probability that this process eventually reaches  $v$ . Our extension of  $\theta$  can be interpreted as  $\theta(v)$  being the probability that this process passes through  $v$ . Formally,  $\theta(\sigma) = 1$ , and  $\theta(u) = 1/\Delta_\sigma \prod_{w \in \langle \sigma, u \rangle} (\Delta_w - 1)$ . It turns out that this choice, slightly changed, works remarkably well, and gives an optimal algorithm in noise conditions that practically match those of our lower bound.

For a vertex  $u \neq \sigma$ , we define  $\beta(u) = \prod_{w \in [\sigma, u)} \Delta_w$ . It is a sort of approximation of  $1/\theta(u)$ , which we prefer for technical convenience. Indeed, for all  $u$ ,  $1/\beta(u) \leq \theta(u)$ . A wonderful property of this  $\beta$  (besides the fact that it gives rise to an optimal algorithm) is that to calculate  $\beta(v)$ , one only needs to know the degrees of the vertices from  $v$  up to the root. It is hard to imagine distributions on leaves that allow us to calculate the probability of being in a subtree without knowing anything about it!

In the walking algorithm, if  $v$  is a candidate for exploration, these nodes must have been visited already and so the algorithm does not need any a priori knowledge of the structure of the tree. The following claim will be soon useful:

**Claim 3.1.** *The following two inequalities hold for every  $c < 1$ :*

$$\sum_{v \in T} \frac{c^{d(v)}}{\beta(v)} \leq \frac{1}{1-c}, \quad \sum_{v \in T} \frac{d(v)c^{d(v)}}{\beta(v)} \leq \frac{c}{(1-c)^2}.$$

*Proof.* To prove the first inequality, follow the same random walk defining  $\theta$ . Starting at the root, at each step choose uniformly at random one of the children of the current vertex. Now, while passing through a vertex  $v$  collect  $c^{d(v)}$  points. No matter what choices are made, the number of points is at most  $1 + c + c^2 + \dots = 1/(1-c)$ . On the other hand,  $\sum_{v \in T} \theta(v)c^{d(v)}$  is the expected number of points gained. The result follows since  $1/\beta(v) \leq \theta(v)$ . The second inequality is derived similarly, using the fact that  $c + 2c^2 + 3c^3 + \dots = c/(1-c)^2$ .  $\square$

For a vertex  $u \in T$  and previously seen advice  $\mathbf{adv}$  define:

$$\mathbf{score}(u) = \frac{2}{3} \log \left( \frac{1}{\beta(u)} \right) - \sum_{w \in \overleftarrow{\mathbf{adv}}(u)} \log(\Delta_w).$$

Algorithm  $\mathbf{A}_{walk}$  keeps track of all vertices that are children of the vertices it explored so far, and repeatedly walks to and then explores the one with highest score according to the current advice, breaking ties arbitrarily. Note that the algorithm does not require prior knowledge of neither the tree's structure, nor the values of  $\Delta$ ,  $q$ ,  $d$  or  $n$ .

### 3.3 Analysis

Recall the definition of Condition  $(\star)$  from Definition 1.2. The next lemma provides a Chernoff's type of bound which is tailored very precisely to our setting. The proof appears in Appendix B.

**Lemma 3.2.** *Consider the random variables  $X_1, \dots, X_\ell$  where  $X_i$  takes values in  $(-\log \Delta_i, 0, \log \Delta_i)$  and has law  $(p_i + \frac{q_i}{\Delta_i}, q_i(1 - \frac{2}{\Delta_i}), \frac{q_i}{\Delta_i})$  for parameters  $p_i, q_i = 1 - p_i$  and  $\Delta_i > 0$ . Assume that Condition  $(\star)$  holds for some  $\varepsilon > 0$ . Then for every integer (either positive or negative)  $m$ , we have:*

$$\Pr \left( \sum_{i=1}^{\ell} X_i \geq m \right) \leq \frac{(1-\varepsilon)^\ell}{e^{\frac{3m}{4}}} \prod_{i=1}^{\ell} \frac{1}{\sqrt{\Delta_i}}.$$



The next theorem states that  $\mathbf{A}_{walk}$  is optimal up to a constant factor for the regime of noise below the threshold. It establishes the first item in Theorem 1.3.

**Theorem 3.3.** *Assume that Condition  $(\star)$  holds for some fixed  $\varepsilon > 0$ . Then  $\mathcal{M}(\mathbf{A}_{walk}) = \mathcal{O}(d\sqrt{\Delta})$ , where the constant hidden in the  $\mathcal{O}$  notation only depends polynomially on  $1/\varepsilon$ .*

*Proof.* Denote the vertices on the path from  $\sigma$  to  $\tau$  by  $\sigma = u_0, u_1, \dots, u_d = \tau$  in order. Denote by  $E_k$  the expected time to reach  $u_k$  once  $u_{k-1}$  is reached. We will show that for all  $k$ ,  $E_k = \mathcal{O}(\sqrt{\Delta})$ , and by linearity of expectation this concludes the proof.

Once  $u_{k-1}$  is visited,  $\mathbf{A}_{walk}$  only goes to some of the nodes that have score at least as high as  $u_k$ . We can therefore bound  $E_k$  from above by assuming we go through all of them, and this expression does not depend on the previous choices of the algorithm and the nodes it saw before seeing  $u_k$ . The length of this tour is bounded by twice the sum of distances of these nodes from  $u_k$ . Hence,

$$E_k \leq 2 \sum_{i=1}^k \sum_{u \in C(u_i)} \Pr(\text{score}(u) \geq \text{score}(u_k)) \cdot d(u_k, u).$$

Where  $C(u_k) = T(u_{k-1}) \setminus T(u_k)$ , and so  $\cup_{i=1}^k C(u_i) = T \setminus T(u_k)$ . Recall that scores are defined so that  $u$  has larger score than  $u_k$ , if the sum of weighted arrows on the path  $\langle u_k, u \rangle$  is at least  $\frac{2}{3} \log(\beta(u)/\beta(u_k))$ . As correct advice on this path point towards  $u_k$ , setting  $m$  to the above value, Lemma 3.2 exactly talks about the probability in question. Denoting  $c = 1 - \varepsilon$ , we get:

$$\begin{aligned} \frac{E_k}{2} &\leq \sum_{i=1}^k \sum_{u \in C(u_i)} \frac{c^{d(u_k, u)-1}}{e^{\frac{3}{4} \cdot \frac{2}{3} \log(\frac{\beta(u)}{\beta(u_k)})}} \sqrt{\prod_{v \in \langle u, u_k \rangle} \frac{1}{\Delta_v}} \cdot d(u_k, u) \\ &= \frac{1}{c} \sum_{i=1}^k \sum_{u \in C(u_i)} \frac{c^{d(u_k, u)}}{\sqrt{\frac{\beta(u)}{\beta(u_k)}}} \sqrt{\frac{\Delta_{u_i}}{\frac{\beta(u_k)}{\beta(u_i)} \cdot \frac{\beta(u)}{\beta(u_i)}}} \cdot d(u_k, u) \\ &\leq \frac{\sqrt{\Delta}}{c} \sum_{i=1}^k c^{d(u_k, u_i)} \sum_{u \in C(u_i)} c^{d(u_i, u)} \frac{\beta(u_i)}{\beta(u)} \cdot (d(u_k, u_i) + d(u_i, u)). \end{aligned}$$

By Claim 3.1, applied in the tree rooted at  $u_i$ , we get:  $\sum_{u \in C(u_i)} \frac{c^{d(u_i, u)} \beta(u_i)}{\beta(u)} < \frac{1}{1-c}$ , as well as  $\sum_{u \in C(u_i)} \frac{c^{d(u_i, u)} \beta(u_i)}{\beta(u)} d(u_i, u) < \frac{c}{(1-c)^2}$ . And so:

$$\frac{E_k}{2} \leq \frac{\sqrt{\Delta}}{c(1-c)} \sum_{i=1}^k c^{d(u_k, u_i)} d(u_k, u_i) + \frac{\sqrt{\Delta}}{(1-c)^2} \sum_{i=1}^k c^{d(u_k, u_i)} \leq \frac{(1+c)\sqrt{\Delta}}{(1-c)^3} \leq \frac{2\sqrt{\Delta}}{\varepsilon^3} = \mathcal{O}(\sqrt{\Delta}),$$

where we again used the equality  $c + 2c^2 + 3c^3 + \dots = c/(1-c)^2$ .  $\square$

### 3.4 Memoryless Algorithms

A walking algorithm does not need to remember more advice and tree structure than the number of its moves. Therefore, since its internal calculations can be somewhat economic,  $\mathbf{A}_{walk}$  can enjoy memory complexity which is polynomial in  $d$  and  $\Delta$ , in expectation. It is interesting to also study the performances of simple algorithms, that do not need to remember any previous action or query. The family of *probabilistic following* algorithms was suggested in [13]. Such an algorithm is characterized by a *listening* parameter  $\lambda$ , indicating the probability that the agent follows the

current advice. It turns out that in trees, such algorithms can be highly efficient when  $q$  is at most roughly  $1/\Delta$  but become exponential above this threshold. More precisely, in Appendix D.2 we prove the following.

**Theorem 3.4.** *There exist positive constants  $c_1, c_2$  and  $c_3$  such that the following holds. If  $q < c_1/\Delta$  then there exists a probabilistic following algorithm that finds the treasure in less than  $c_2d$  expected steps. On the other hand, if  $q > c_3/\Delta$  then for any probabilistic following strategy the move complexity on a complete  $\Delta$ -ary tree is exponential in the depth of the tree.*

## 4 An Almost Optimal Query Algorithm

In this section, we consider the query setting, where the tree structure is known to the algorithm. Our aim is to prove the second part of Theorem 1.3, that is, we wish to prove the following.

**Theorem 4.1.** *There exists a deterministic algorithm  $A_{\text{query}}$  that satisfies the following. Assume that Condition  $(\star)$  holds for some fixed  $0 < \varepsilon < 1/2$ , then  $\mathcal{Q}(A_{\text{query}}) = \mathcal{O}(\sqrt{\Delta} \log n)$ . The  $\mathcal{O}(\cdot)$  notation only hides a polynomial dependency on  $1/\varepsilon$ .*

First note that on the line, several algorithms achieving the optimal number of queries  $\mathcal{O}(\log n)$  are known in the setting of noise that can be resampled [3, 9]. These can be easily adapted to our setting by replacing a query to a node  $u \in [n]$  by querying the nearest node to  $u$  that has not yet been queried. Such an obvious adaptation fails on general trees, since there, the information an algorithm gets when querying a node  $v \neq u$  is, in general, very different from querying  $u$  again.

One natural approach to achieve the desired bound on a general tree  $T$ , is to decompose it into separators, build the corresponding separator-tree  $T^{\text{sep}}$  with the advice inherited from  $T$ , and run  $A_{\text{walk}}$  over  $T^{\text{sep}}$ . Since the depth of  $T^{\text{sep}}$  is  $\mathcal{O}(\log n)$ , the hope is that the  $\mathcal{O}(\sqrt{\Delta} \log n)$  bound for the query complexity would follow directly from the move complexity of  $A_{\text{walk}}$ . Unfortunately, however, it is not so obvious how to interpret in  $T^{\text{sep}}$  the advice of a node  $v \in T$ . To see why, consider for example the line  $1, 2, \dots, n$ . The root of  $T^{\text{sep}}$  is  $n/2$ , and let's say that  $\text{adv}(n/2) = n/2 - 1$ . We interpret this advice in  $T^{\text{sep}}$  as pointing to  $n/4$ . Now, if  $\text{adv}(n/4) = n/4 + 1$ , it is not clear how to interpret this advice in  $T^{\text{sep}}$ , should  $n/4$  point back to  $n/2$ , or to its child  $3n/8$ .

*Proof of Theorem 4.1.* Instead of the aforementioned separator based approach, we use another decomposition. Starting with a tree  $T$ , we fix a root arbitrarily and contract every path to obtain a tree  $T'$ . More specifically, a *pure path* is a maximal path  $P = (u_1, u_2, \dots, u_k)$ , where  $k > 3$ , and all internal nodes  $u_i$ ,  $1 < i < k$ , have degree 2. Each pure path  $P$  in  $T$  is contracted into a path of three nodes,  $P' = (u_1, u_2, u_k)$ . We call the node  $u_2$  the *representative point* of the path. It is easy to see that the depth of  $T'$  is  $\mathcal{O}(\log n)$ .

Let us denote by  $\tau'$  the node in  $T'$  corresponding to  $\tau \in T$ . Namely, if  $\tau$  belongs to an internal node of a pure path in  $T$  then  $\tau'$  is the representative point of that path and otherwise,  $\tau' = \tau$ . It is easy to see that the advice for every node  $u \in T'$ , where  $u \neq \tau'$ , behaves as if  $\tau'$  is the treasure of  $T'$ . Hence, if we run  $A_{\text{walk}}$  on  $T'$ , it is guaranteed to find  $\tau'$  in  $\mathcal{O}(\sqrt{\Delta} \log n)$  expected walking steps (which correspond to queries here). In particular, if  $\tau' = \tau$  then running  $A_{\text{walk}}$  on  $T'$  finds it fast. Otherwise,  $\tau$  is an internal node of some pure path, and is not its representative point. In this case,  $A_{\text{walk}}$  has no way of detecting that it found  $\tau'$ , as it only learns it found  $\tau$  when it queries it.

We denote by  $R \subseteq T'$  the set of representative points of pure paths in  $T$ . Our algorithm  $A_{\text{query}}$  runs in consecutive phases. A phase consists of four consecutive stages: First, it runs  $A_{\text{walk}}$  on the contracted tree  $T'$  for  $\mathcal{O}(\log n)$  queries, continuing from where it stopped in the previous phase. Second, it identifies the node  $u$  of highest score among the visited ones of  $R$ , that was not processed

already in previous phases. Third, it executes a protocol to decide whether or not  $\tau$  belongs to the pure path corresponding to  $u$ . Finally, if the answer to this decision protocol is “yes” then  $A_{query}$  runs a line algorithm on this path. In parallel to all the above,  $A_{query}$  runs an exhaustive search algorithm therefore bounding the worst case running time by  $\mathcal{O}(n)$ .

To complete the description of  $A_{query}$  it remains to describe the decision protocol. Given a path, we want to identify with high probability in  $n$ , whether the treasure is there, while spending  $\mathcal{O}(\log n)$  queries. This is simple: For a constant  $C$ , if the path is already of size  $C \log n$ , we just query it all, and otherwise, we query the  $C \log n$  nodes on both extremities. We output “yes” iff on both sides, the majority of pointers point towards the other side. We choose  $C$  big enough, so that by Chernoff’s bound, the probability of a fixed path being a “false positive” or “false negative” is at most  $1/n^2$ . Taking a union bound over all paths we get that the probability of the path-decision protocol errs at any phase is at most  $1/n$ . Note that the expected running time of each phase is  $\mathcal{O}(\log n)$ , conditioning on that the decision protocol never makes a mistake.

After  $\mathcal{O}(\sqrt{\Delta})$  phases in expectation  $\tau'$  is “found” and after  $\mathcal{O}(\sqrt{\Delta})$  extra phases it becomes the highest ranked node of  $R$ , for which the corresponding path has not been searched. This is because, as we prove below, there are in expectation  $\mathcal{O}(\sqrt{\Delta})$  nodes in  $R$  which have score at least as high as  $\tau'$ . Hence, after a total of  $\mathcal{O}(\sqrt{\Delta})$  phases in expectation, the path where  $\tau$  lies is searched.

All that is left is to bound the number of nodes in  $R$  which have score at least as high as  $\tau'$ . We first note that unlike the situation in  $A_{walk}$ , here we compare vertices that we have actually read the advice on, and this must go into our accounting by our definition of the score of a vertex.

We first show that considering the end nodes of the path can increase the value of  $\Pr(\text{score}(u) \geq \text{score}(\tau'))$ , but only by very little. Recall that the advice at a vertex never counts towards its own score, and that the score of the vertex involves subtracting the weight of the advice pointing away from it. Therefore, the advice at  $\tau'$  can, in the worst case, point towards  $u$ , and add nothing to  $\text{score}(u)$ . Therefore,  $\text{score}(u)$  can never increase. The advice at  $u$  can point away from  $\tau$  and so decrease  $\log(2)$  from  $\text{score}(\tau')$ , since  $u$  is of degree 2. In total, this means, that for  $u$  to be checked before  $\tau'$ , it must satisfy that the sum of advice on  $\langle u, \tau' \rangle$  is at least  $\frac{2}{3} \log(\beta(u)/\beta(\tau')) - \log(2)$ .

Let us consider vertices that are descendants of  $\tau'$  in  $T'$ , denoted  $T'(\tau')$ . The expected number of nodes  $u \in R \cap T'(\tau')$  which have higher score than  $\tau'$  is at most:

$$\sum_{u \in T'(\tau')} \Pr \left( \sum_{i=1}^{d(u)-1} X_i \geq \frac{2}{3} \log(\beta(u)) - \log(2) \right) \leq \sum_{u \in T'(\tau')} \frac{e^{\frac{3}{4} \log(2)} (1 - \varepsilon)^{d(u)-1}}{\beta(u)} < \frac{4}{\varepsilon} = \mathcal{O}(1),$$

where  $\beta(u)$  and  $d(u)$  are defined as if  $\tau'$  is the root (in particular  $\beta(\tau') = 1$ ), and  $X_i$  are defined as in Lemma 3.2. In the last inequality we used Claim 3.1.

If we now consider the set of vertices in  $R$  that are not descendants of  $\tau'$ , we can follow the exact calculation of Theorem 3.3, except that the sum of weighted arrows on the path  $\langle \tau', u \rangle$  should again be at least  $\frac{2}{3} \log(\beta(u)/\beta(\tau')) - \log(2)$  (i.e., change of  $\log(2)$  from the original quantity). Exactly as above, this changes the resulting sum by less than a factor of 2. We therefore get that, in expectation, there are at most  $\mathcal{O}(\sqrt{\Delta})$  vertices in  $R$  that will be checked before  $\tau'$ , as required.  $\square$

## 5 Open Problems

We leave several directions for further research. Extending our results to general graphs is highly intriguing. Even though the likelihood of a node being the target (when the treasure is placed uniformly at random in one of them) can still be computed in principle, it is not so easy to compare two nodes as in Theorem 3.3 because there may be more than a single path between them.

In a limited regime of noise, we believe that memoryless strategies might very well be efficient also on general graphs, and we pose the following conjecture. Proving it will require the use of tools from the theory of RWRE, which are currently lacking in the context of general graph topologies.

**Conjecture 5.1.** *There exists a probabilistic following algorithm that finds the treasure in expected linear time on any undirected graph assuming  $q < c/\Delta$  for a small enough  $c > 0$ .*

Finally, we note that the model may also be modified to include multiple treasures.

# Appendix

## A Lower bounds

### A.1 An Exponential Lower Bound Above the Threshold: Proof of Lemma 2.1

For the lower bound, assume the algorithm is given the advice  $\mathbf{adv}$  for all the internal nodes for free. By Yao's principle, instead of taking the worst case placement of the treasure for a randomized algorithm, we obtain a lower bound by considering only deterministic algorithms when the treasure is placed uniformly at random at one of the leaves.

It turns out that the behavior of an optimal algorithm is explicit in this setting: It sorts the leaves according to  $\Pr(\cdot | \mathbf{adv})$  (Claim A.1) and tries them in this order. It is then easy to check that this order corresponds to ranking nodes by how many arrows point to them (Claim A.2). The expected number of nodes which are higher than the treasure in this ordering is therefore a lower bound for this algorithm, and thus for all algorithms.

Let  $\mathcal{L}$  be the set of leaves. For a given leaf  $u \in \mathcal{L}$  and an advice configuration  $\mathbf{adv}$ , let  $C(\mathbf{A}, \mathbf{adv}, u)$  be the cost (number of queries) of  $\mathbf{A}$  when the advice is equal to  $\mathbf{adv}$  and the treasure is located at  $u$ . We also define the cost  $C(\mathbf{A}, u)$  of an algorithm  $\mathbf{A}$  when the treasure  $\tau$  is located at  $u$  to be the expected cost of  $\mathbf{A}$  before finding  $\tau$  where the expectation is over advice setting. That is:

$$C(\mathbf{A}, u) = \sum_{\mathbf{adv}} C(\mathbf{A}, \mathbf{adv}, u) \Pr(\mathbf{adv} | u).$$

In our setting, the expected number of queries of  $\mathbf{A}$  is:

$$C(\mathbf{A}) = \sum_{u \in \mathcal{L}} \Pr(u) \sum_{\mathbf{adv}} C(\mathbf{A}, \mathbf{adv}, u) \Pr(\mathbf{adv} | u).$$

**Claim A.1.** *The algorithm  $\mathbf{A}$  that tries the locations  $u$  in the order given by  $\Pr(u | \mathbf{adv})$ , i.e., the most likely  $u$  is tried first and the least likely tried last, minimizes  $C(\mathbf{A})$ .*

*Proof.* We can write

$$C(\mathbf{A}) = \sum_{\mathbf{adv}} \Pr(\mathbf{adv}) \sum_{u \in \mathcal{L}} C(\mathbf{A}, \mathbf{adv}, u) \Pr(u | \mathbf{adv}),$$

where it is understood that  $\Pr(\mathbf{adv})$  is the marginal of  $\Pr(\mathbf{adv}, u)$  with respect to the advice. Note that the term  $\Pr(u | \mathbf{adv})$ , standing for the probability of  $u$  holding the treasure given that the advice configuration is  $\mathbf{adv}$ , is only defined because we assume the treasure is placed according to a known distribution (uniform in our case). For a fixed advice setting  $\mathbf{adv}$ , it follows from the *rearrangement inequality* that  $\sum_{u \in \mathcal{L}} C(\mathbf{A}, \mathbf{adv}, u) \Pr(u | \mathbf{adv})$  is minimized when  $C(\mathbf{A}, \mathbf{adv}, u)$  and  $\Pr(u | \mathbf{adv})$  are sorted in the same order with respect to  $u$ . This corresponds to algorithm  $\mathbf{A}$  trying the locations  $u$  in the order given by  $\Pr(u | \mathbf{adv})$ , which is exactly the statement of the claim. Hence, since we assume that all advice is known, the algorithm we have just described is feasible, and, in fact, optimal. Moreover, its query complexity is at least 1 plus the expected number of nodes which are strictly more likely than the treasure, where the expectation is taken over the randomness of the advice.  $\square$

It only remains to check that a node  $u$  is more likely than  $\tau$  given an advice setting  $\mathbf{adv}$  iff more arrows point to  $u$  than  $\tau$ . This will conclude the proof of Lemma 2.1 and hence of the exponential lower bound in Theorem 1.1.

**Claim A.2.** For two leaves  $u, v \in \mathcal{L}$ , and advice configuration  $\mathbf{adv}$ ,  $\Pr(u | \mathbf{adv}) > \Pr(v | \mathbf{adv})$  if and only if there is more advice pointing towards  $u$  than advice pointing towards  $v$ .

*Proof.* Recall that, by definition of the model

$$\Pr(\mathbf{adv} | \tau = u) = \left(p + \frac{q}{\Delta}\right)^{|\overrightarrow{\mathbf{adv}}(u)|} \left(q\left(1 - \frac{1}{\Delta}\right)\right)^{|\overleftarrow{\mathbf{adv}}(u)|},$$

In our regime it will always be the case that  $p + \frac{q}{\Delta} > q\left(1 - \frac{1}{\Delta}\right)$ , simply because we assume  $q < p$ . Hence  $\Pr(\mathbf{adv} | \tau = u)$  is an increasing function of  $|\overrightarrow{\mathbf{adv}}(u)|$ .

Since  $\tau$  is placed uniformly at random, it follows from Bayes rule that  $\Pr(\mathbf{adv} | \tau = u) \propto \Pr(\tau = u | \mathbf{adv})$ . The symbol  $\propto$  indicates that we omit the renormalizing factor. Hence, we obtain that  $\Pr(\tau = u | \mathbf{adv}) > \Pr(\tau = v | \mathbf{adv})$  if and only if  $|\overrightarrow{\mathbf{adv}}(u)| > |\overrightarrow{\mathbf{adv}}(v)|$ .  $\square$

## A.2 A Lower Bound for the Move Complexity

**Observation A.3.** For any  $\Delta$  and  $d$ , there exists a tree of depth  $d$  and maximal degree at most  $\Delta$  for which any search algorithm  $A$  has move complexity  $\mathcal{M}(A) = \Omega(dq\Delta)$ . In particular, when  $q \sim 1/\sqrt{\Delta}$ , we have  $\mathcal{M}(A) = \Omega(d\sqrt{\Delta})$ .

*Proof.* To see why the observation holds consider the caterpillar tree, composed of a path of length  $n/\Delta$  with each of its nodes being the center of a star graph of degree  $\Delta$ . Assume that the agent starts at one of the end sides of the path and the treasure at distance  $d$  on the caterpillar spine. Recall that we assume that the algorithm does not know the tree structure. On expectation,  $\Omega(dq)$  nodes will point at an incorrect neighbor, and to pass from any of those to the next node on the path, will require the agent to perform  $\Omega(\Delta)$  trials in expectation.  $\square$

## B Proof of the Chernoff Estimate (Lemma 3.2)

For any  $s \in \mathbb{R}$ , we have:

$$\begin{aligned} \Pr\left(\sum_{i=1}^{\ell} X_i \geq m\right) &= \Pr\left(e^s \sum_{i=1}^{\ell} X_i \geq e^{sm}\right) \leq \frac{\mathbb{E}\left[e^{s \sum_{i=1}^{\ell} X_i}\right]}{e^{sm}} = \frac{\prod_i \mathbb{E}\left[e^{s X_i}\right]}{e^{sm}} \\ &= \frac{1}{e^{sm}} \prod_{i=1}^{\ell} \left( \frac{p_i + \frac{q_i}{\Delta_i}}{e^{\log(\Delta_i)s}} + q_i \left(1 - \frac{2}{\Delta_i}\right) + \frac{q_i}{\Delta_i} e^{\log(\Delta_i)s} \right) \\ &\leq \frac{1}{e^{sm}} \prod_{i=1}^{\ell} \left( \frac{1}{\Delta_i^s} + q_i + q_i \Delta_i^{s-1} \right). \end{aligned}$$

We take  $s = \frac{3}{4}$ , and get:

$$\Pr\left(\sum_{i=1}^{\ell} X_i \geq m\right) \leq \frac{1}{e^{\frac{3m}{4}}} \prod_{i=1}^{\ell} \left( \Delta_i^{-\frac{3}{4}} + q_i + q_i \Delta_i^{-\frac{1}{4}} \right) \leq \frac{1}{e^{\frac{3m}{4}}} \prod_{i=1}^{\ell} \frac{1 - \varepsilon}{\sqrt{\Delta_i}}$$

Where for the last step we used Condition  $(\star)$  which says:

$$q_i < \frac{1 - \varepsilon - \Delta_i^{-\frac{1}{4}}}{\sqrt{\Delta_i} + \Delta_i^{\frac{1}{4}}}$$

## C Another variant on the greedy Bayesian search that fails

As mentioned at the end of Section 3.1, when our tree is a full tree, choosing  $\theta$  to be the uniform distribution over the leaves results in an efficient algorithm with respect to the worst case placement of the treasure. Trying to tackle more general trees, perhaps the most natural a priori distribution is the uniform one over the nodes of the tree. As our technical presentation accommodates only distributions on leaves, we take  $\theta$  to be uniform over the leaves only, and remark that the same result we get here applies to the former case.

Perhaps surprisingly, we show that this variant may take exponentially many queries before finding the treasure no matter what  $q$  is. The instance we consider is a complete  $\Delta$ -ary tree of depth  $D$ , except for one child of the root, which is turned into a leaf, trimming its  $(\Delta - 1)^{D-1}$  descendants. We consider the case that this particular child is in fact the treasure location  $\tau$ .

Recall from Section 3.1 that  $\text{score}(u) > \text{score}(\tau)$  iff:

$$\sum_{w \in \langle u, \tau \rangle \cap \overrightarrow{\text{adv}}(u)} \log(\Delta_w) - \sum_{w \in \langle u, \tau \rangle \cap \overrightarrow{\text{adv}}(\tau)} \log(\Delta_w) > \frac{2}{3} \log \left( \frac{\theta(\tau)}{\theta(u)} \right), \quad (1)$$

where  $\theta(u)$  is now understood as the ratio between the number of leaves in  $T(u)$  divided by the total number of leaves in  $T$ , as opposed to the total number of leaves if the tree was a complete tree.

In particular, consider any node  $u$  at distance  $a \cdot D$  from the root for some  $a < 2/5$ . This node  $u$  owns a tree  $T(u)$  of size  $(\Delta - 1)^{D(1-a)}$ , hence  $\theta(u) \sim (\Delta - 1)^{-a}$ . In contrast  $\theta(\tau) = (\Delta - 1)^{-D}$ . Therefore,

$$\frac{2}{3} \log \left( \frac{\theta(\tau)}{\theta(u)} \right) = -\frac{2}{3} D(1-a) \log(\Delta),$$

where we write  $\log(\Delta)$  in place of  $\log(\Delta - 1)$  as it does not change the nature of the result, only the choice of the constant  $2/5$ . On the other hand there are only  $a\Delta$  nodes on the path  $(u, \tau)$  so the left side of equation 1 is always greater than  $-aD \log \Delta$ . In other words if  $a < 2/5$  then then  $-\frac{2}{3} D(1-a) \log \Delta < -aD \log \Delta$ , and any node  $u$  at depth  $a \cdot D$  has a better score than  $\tau$ , regardless of the advice on the path  $\langle \tau, u \rangle$  which means that our algorithm needs  $(\Delta - 1)^{2/5D}$  steps at least.

## D Memoryless Algorithms and the Semi-Adversarial Model

In this section we present our results on the memoryless algorithms described in the introduction. As mentioned, such algorithms can perform well also in a more difficult semi-adversarial setting. Before we present these algorithms let us first describe formally the semi-adversarial variant.

**Definition D.1** (The Semi-Adversarial Model). As in the purely-probabilistic Noisy Advice Model, each node is chosen to be *faulty* with probability  $q$ , and otherwise it is *sound*. Also, similarly to the original model, a sound vertex always points at its correct neighbors. However, in the semi-adversarial model, a faulty node  $u$  no longer points at a neighbor chosen uniformly at random, and instead, the neighbor  $w$  which such a node points at is chosen by an adversary. Importantly, for each node  $u$ , the adversary must specify its potentially faulty advice  $w$ , before it is known which nodes will be faulty. In other words, first, the adversary specifies the faulty advice  $w$  for each node  $u$ , and then the environment samples which node is faulty and which is sound.

### D.1 Lower Bound in the Semi-Adversarial Variant

The following result implies that if  $q > 1/\Delta$  then any algorithm must have query complexity and move complexity being exponential in the depth  $D$  (or polynomial in  $n$ ).

**Theorem D.2.** *Consider an algorithm in the semi-adversarial model. On the complete  $\Delta$ -ary tree of depth  $D$ , the expected number of queries to find the treasure is  $\Omega((q\Delta)^D)$ . The lower bound holds even if the algorithm has access to the advice of all internal nodes in tree.*

*Proof.* Consider the complete  $\Delta$ -ary tree and assume that the treasure is located at a leaf. The adversary behaves as follows. For any advice it gets a chance to manipulate, it would always make it point towards the root. With probability  $q^D$  the adversary gets to choose all the advice on the path between the root and the treasure. Any other advice points towards the root as well (either because it was correct to begin with or because it was set by the adversary). Hence with probability  $q^D$  the tree that the algorithm sees is the same regardless of the position of the treasure. It follows from Observation 2.2 that the time to find the treasure can only be linear in the number of leaves which is  $\Omega(\Delta^D)$ .  $\square$

## D.2 Probabilistic Following Algorithms

Recall that a *Probabilistic Following* (PF) algorithm is specified by a *listening* parameter  $\lambda \in (0, 1)$ . At each step, the algorithm “listens” to the advice with probability  $\lambda$  and takes a uniform random step otherwise. The first item in the next theorem states that if the noise parameter is smaller than  $c/\Delta$  for some small enough constant  $0 < c < 1$ , then there exists a listening parameter  $\lambda$  for which Algorithm PF achieves  $\mathcal{O}(d)$  move complexity. Moreover, this result holds also in the semi-adversarial model. Hence, together with Theorem D.2, it implies that in order to achieve efficient search, the noise parameter threshold for the semi-adversarial model is roughly  $1/\Delta$ .

**Theorem D.3.** 1. *Assume that  $q < 1/(10\Delta)$ . Then PF with parameter  $\lambda \in [\frac{1}{3}, \frac{2}{3}]$  finds the treasure in less than  $100d$  expected steps, even in the semi-adversarial setting.*

2. *Consider the complete  $\Delta$ -ary tree and assume that  $q > 10/\Delta$ . Then for any choice of  $\lambda$  the hitting time of the treasure by PF is exponential in the depth of the tree, even assuming the faulty advice is drawn at random.*

*Proof.* Our plan is to show that the expected time to make one step in the correct direction is  $\mathcal{O}(1)$ , from any starting node. Conditioning on the advice setting, we make use of the Markov property to relate these elementary steps to the total travel time. The main delicate point in the proof stems from dealing with two different sources of randomness. Namely the randomness of the advice and that of the walk itself.

It will be convenient to picture the tree as rooted at the target node  $\tau$ . For any node  $u$  in the tree, we denote by  $u'$  the parent of  $u$  with respect to the treasure. With this convention, correct advice at a node  $u$  points at  $u'$ , while incorrect advice points at one of its children. The fact the walk moves on a tree means that for a given advice setting, the expected (over the walk) time it takes to reach  $u'$  from  $u$  can be written conveniently as a product of a variable involving the advice at  $u$  only and the advice on the set of  $u$ 's descendants (the two being independent).

We denote by  $t(u)$  the time it takes to reach node  $u$ . Note that we need to be cautious with what we mean by average symbols such as  $\mathbb{E}$ . Indeed there are two sources of randomness, the first being the randomness used in drawing the advice and the second being the randomness used in the walk itself. We write  $\mathbb{E}$  for averaging over the advice, while we use  $E_u$  to denote expectation over the walk, conditioning on  $u$  being the starting node. As a remark, observe that  $E_u(t(v))$  depends on the advice configuration, it is a random variable with respect to the advice, while  $\mathbb{E}E_u(t(v))$  really is just a number.

The following is the central lemma of this section.

**Lemma D.4.** *Assume that  $q < 1/(10\Delta)$ , and  $\lambda \in [\frac{1}{3}, \frac{2}{3}]$ . Then for all nodes  $u$ ,  $\mathbb{E}E_u t(u') \leq 100$ . The result holds also in the semi-adversarial model.*



Let us now see how we can conclude the proof of the first item in Theorem D.3, given the lemma. Consider a designated source  $\sigma$ . Let us denote by  $\sigma = u_d, u_{d-1}, \dots, u_0 = \tau$  the nodes on the path from  $\sigma$  to  $\tau$ . Let  $\delta_i$  be the random variable indicating the time it takes to reach  $u_{i-1}$  after  $u_i$  has been visited for the first time. With these notations, the time to reach  $\tau$  from  $\sigma$  is precisely  $\sum_{i=1}^{d(\sigma, \tau)} \delta_i$ . Hence, the expected time to reach  $\tau$  from  $\sigma$  is, by linearity of expectation:

$$\sum_{i=1}^{d(\sigma, \tau)} \mathbb{E}[E_\sigma \delta_i] .$$

Conditioning on the advice setting, the process is a Markov chain and we may write

$$E_\sigma \delta_i = E_{u_i} t(u_{i-1}).$$

Taking expectations over the advice ( $\mathbb{E}$ ), under the assumptions of Lemma D.4, it follows that  $\mathbb{E}(E_\sigma \delta_i) \leq 100$ , for every  $i \in [d(\sigma, \tau)]$ . And this immediately implies a bound of  $100 \cdot d(\sigma, \tau)$ .

*Proof of Lemma D.4.* We start with partitioning the nodes of the tree according to their distance from the root  $\tau$ . More precisely, for  $i = 1, 2, \dots, D$ , where  $D$  is the depth of the tree, let

$$\mathcal{L}_i := \{u \in T : d(u, \tau) = i\} .$$

The nodes in  $\mathcal{L}_i$  are referred to as *level- $i$*  nodes. We treat the statement of the lemma for nodes  $u \in \mathcal{L}_i$  as an induction hypothesis, with  $i$  being the induction parameter. The induction goes backwards, meaning we assume the assumption holds at level  $i+1$  and show it holds at level  $i$ . The case of the maximal level (base case for the induction) is easy since, at a leaf the walk can only go up and so if  $u$  is a leaf  $\mathbb{E}E_u(t(u')) = 1 < 100$ .

Assume now that  $u \in \mathcal{L}_i$ . We first condition on the advice setting. A priori,  $E_u t(u')$  depends on the advice over the full tree, but in fact it is easy to see that only advice at layers  $\geq i$  matter. Recall from Markov Chain theory that an *excursion* to/from a point is simply the part of the walk between two visits to the given point. We denote  $L_u$  the average (over the walk only) length of an excursion from  $u$  to itself that does not go straight to  $u'$  and we write  $N_u$  to denote the expected (over the walk only) number of excursions before going to  $u'$ . We also refer to this number as a number of *attempts*. Note that  $N_u$  can be 0 if the walk goes directly to  $u'$  without any excursion. We decompose  $t(u')$  in the following standard way, using the Markov property

$$E_u t(u') = 1 + L_u \cdot N_u. \tag{2}$$

Indeed the expectation  $E_u t(u')$  can be seen as the expectation (over the walk) of  $1 + \sum_{i=1}^T Y_i$  where the  $Y_i$ 's are the lengths of each excursion from  $u$  and  $T$  is the (random) number of such excursions before hitting  $u'$ . The term 1+ accounts for the step from  $u$  to  $u'$ . Note that  $\{T \geq t\}$  is independent of  $Y_1, \dots, Y_t$  and so using Wald's identity we have that  $E_u t(u') = 1 + E_u T \cdot E_u Y_1$ . The term  $E_u T$  is equal to  $N_u$  (by definition) while  $E_u Y_1$  is equal to  $L_u$  (by definition).

We now want to average equality (2), which is only an average over the walk, by taking the expectation over all advice in layers  $\geq i$ . To this aim, note that  $L_u$  can be written as

$$L_u = 1 + \sum_{v \neq u', v \sim u} p_{u,v} E_v t(u),$$

where we write  $u \sim v$  when  $u$  and  $v$  are neighbors in the tree and  $p_{u,v}$  is the probability to go straight from  $u$  to  $v$  given the advice setting. Note that  $E_v t(u)$  depends on the advice at layers

$\geq i + 1$  only, if we start at a node  $v \in \mathcal{L}_{i+1}$ , while both  $p_{u,v}$  and  $N_u$  depend only on the advice at layer  $= i$  of the tree, by assumption on the model. This is true also in the semi-adversarial model. Hence when we average, we can first average over layers  $> i$  to obtain, denoting  $\mathbb{E}^{>i}$ , the expectation over the layers  $> i$ ,

$$\begin{aligned}\mathbb{E}^{>i} E_u t(u') &= 1 + \left( 1 + \sum_{v \neq u', v \sim u} p_{u,v} \mathbb{E}^{>i} E_v t(u) \right) N_u, \\ &= 1 + \left( 1 + \sum_{v \neq u', v \sim u} p_{u,v} \mathbb{E} E_v t(u) \right) N_u.\end{aligned}\tag{3}$$

and using the fact that,

$$\sum_{v \neq u'} p_{u,v} \leq 1,\tag{4}$$

together with the induction assumption at rank  $i + 1$ , we obtain

$$\mathbb{E}^{>i} E_u t(u') \leq 1 + (1 + 100) N_u.$$

From now on we replace 100 by a parameter  $\kappa > 0$ , for mere aesthetic reasons. Averaging over the layer  $i$  of advice we obtain

$$\mathbb{E} E_u t(u') \leq 1 + (1 + \kappa) \mathbb{E} N_u.$$

It only remains to analyze the term  $\mathbb{E} N_u$ . If the advice at  $u$  is correct, which happens with probability  $p$ , then the number of attempts follows a (shifted by 1) geometric law with parameter  $\lambda + \frac{(1-\lambda)}{\Delta}$ . In words, when the advice points to  $u'$  which happens with probability  $p + \frac{q}{\Delta}$ , the walker can go to the correct node either because she listens to the advice, which happens with probability  $\lambda$ , or because she did not listen, but still took the right edge, which happens with probability  $\frac{(1-\lambda)}{\Delta}$ . Similarly, when the advice points to a node  $\neq u'$ , which happens with probability  $q(1 - \frac{1}{\Delta})$ , then  $N_u$  follows a geometric law (shifted by 1) with parameter  $\frac{(1-\lambda)}{\Delta}$ . The conclusion is that

$$\begin{aligned}\mathbb{E} N_u &= \left[ \left( p + \frac{q}{\Delta} \right) \left( \frac{1}{\lambda + \frac{(1-\lambda)}{\Delta}} - 1 \right) + q \left( 1 - \frac{1}{\Delta} \right) \left( \frac{\Delta}{1 - \lambda} - 1 \right) \right] \\ &\leq \left( \frac{2}{1 + \lambda} + q \frac{\Delta}{1 - \lambda} - 1 \right).\end{aligned}\tag{5}$$

And so it follows that

$$\mathbb{E} E_u t(u') \leq 1 + (1 + \kappa) \cdot \left( \frac{2}{1 + \lambda} + q \frac{\Delta}{1 - \lambda} - 1 \right).$$

Where we used the fact that  $\Delta \geq 2$  in the last inequality, and we bounded  $p + \frac{q}{\Delta}$  by 1. Hence if  $q\Delta < 0.1$  and we choose  $\lambda \in [\frac{1}{3}, \frac{2}{3}]$  (for instance, we made no attempt in optimizing these constants), we see that  $\mathbb{E} N_u < 0.8$ . Indeed, by convexity, it is enough to check the inequality at the two endpoints. When  $\lambda = 1/3$ , we obtain

$$\mathbb{E} N_u \leq \frac{3}{2} - 1 + \frac{3}{20} < \frac{8}{10},$$

and when  $\lambda = 2/3$ , we obtain

$$\mathbb{E}N_u \leq \frac{6}{5} - 1 + \frac{3}{10} < \frac{8}{10}.$$

Hence it follows that

$$\mathbb{E}E_ut(u') \leq 1 + 0.8(1 + \kappa) < \kappa.$$

The last inequality holds by choice of  $\kappa = 100$ . By our (backwards) induction, we have just shown that, if  $q < \frac{1}{10\Delta}$  and we set  $\lambda \in [\frac{1}{3}, \frac{2}{3}]$  then for all nodes  $u$  in the tree

$$\mathbb{E}E_ut(u') < 100.$$

This concludes the proof of Lemma D.4 and hence also of the first part of Theorem D.3.  $\square$

Let us explain how the lower bound in the second part of Theorem D.3 is derived in the case that  $q\Delta > 10$ . We assume we are in a complete  $\Delta$ -ary tree. Using the same notations as previously and Equation 5,

$$\mathbb{E}(N_u) \geq q\Delta \left(1 - \frac{1}{\Delta}\right) \frac{1}{1 - \lambda} - 1 \geq \frac{10(1 - \frac{1}{\Delta})}{1 - \lambda} - 1 \geq 10 \left(1 - \frac{1}{\Delta}\right) - 1 \geq 3,$$

for any choice of  $\lambda$ , since  $\Delta \geq 2$ . We proceed very similarly, by induction, and use Equality (3) together with the previous bound on  $\mathbb{E}(N_u)$  to obtain that for any node on layer  $i$ ,  $u$  with parent  $u'$ ,  $\mathbb{E}E_ut(u') \geq 1 + 3 \min_{v \in \mathcal{L}_{i+1}} \mathbb{E}E_vt(v')$ , so in particular  $\min_{u \in \mathcal{L}_i} \mathbb{E}E_ut(u') \geq 1 + 3 \min_{v \in \mathcal{L}_{i+1}} \mathbb{E}E_vt(v')$ . The expected hitting time of the target  $\tau$ , even starting at one of its children is therefore of order  $\Omega(3^D)$ .  $\square$

**Remark D.5.** *Note that the proof uses crucially the tree structure and does not extend to general graphs straightforwardly. Specifically, on a tree there is a single path from  $\sigma$  to  $\tau$  and so the points  $u_i$  are uniquely defined, they are not random. Moreover an excursion from a node  $u$  at Layer  $i$  that does not visit its parent can only remain in layers  $\geq i$ . This was used when we said that  $E_vt(u)$  depends only on the advice at layers  $\geq i$ , if we start at a node  $v \in \mathcal{L}_i$ .*

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